Universal localization of triangular matrix rings

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Abstract

If R is a triangular 2×2 matrix ring, the columns, P and Q, are f.g. projective R-modules. We describe the universal localization of R which makes invertible an R-module morphism $\sigma : P \to Q$, generalizing a theorem of A.Schofield. We also describe the universal localization of R-modules.

1 Introduction

Suppose R is an associative ring (with 1) and $\sigma : P \to Q$ is a morphism between finitely generated projective R-modules. There is a universal way to localize R in such a way that σ becomes an isomorphism. More precisely there is a ring morphism $R \to \sigma^{-1}R$ which is universal for the property that

$$\sigma^{-1}R \otimes_R P \xrightarrow{1 \otimes \sigma} \sigma^{-1}R \otimes_R Q$$

is an isomorphism (Cohn [7, 9, 8, 6], Bergman [4, 5], Schofield [17]). Although it is often difficult to understand universal localizations when R is non-commutative[1] there are examples where elegant descriptions of $\sigma^{-1}R$ have been possible (e.g. Cohn and Dicks [10], Dicks and Sontag [11, Thm. 24], Farber and Vogel [12] Ara, González-Barroso, Goodearl and Pardo [1, Example 2.5]). The purpose of this note is to describe and to generalize some particularly interesting examples due to A.Schofield [17, Thm. 13.1] which have application in topology (e.g. Ranicki [16, Part 2]).

We consider a triangular matrix ring $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ where A and B are associative rings (with 1) and M is an (A, B)-bimodule. Multiplication in R is given by

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix}$$

for all $a, a' \in A$, $m, m' \in M$ and $b, b' \in B$. The columns $P = \begin{pmatrix} A \\ 0 \end{pmatrix}$ and $Q = \begin{pmatrix} M \\ B \end{pmatrix}$ are f.g. projective left R-modules with

$$P \oplus Q \cong R.$$

The general theory of triangular matrix rings can be found in Haghany and Varadarajan [13, 14].

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[1] If R is commutative one obtains a ring of fractions; see Bergman [5, p.68].

We shall describe in Theorem 2.4 the universal localization $R \to \sigma^{-1}R$ which makes invertible a morphism $\sigma : P \to Q$. Such a morphism can be written $\sigma = \begin{pmatrix} j \\ 0 \end{pmatrix}$ where $j : A \to M$ is a morphism of left A-modules. Examples follow, in which restrictions are placed on A, B, M and σ. In particular Example 2.8 recovers Theorem 13.1 of Schofield [17]. We proceed to describe the universal localization $\sigma^{-1}N = \sigma^{-1}R \otimes_R N$ of an arbitrary left module N for the triangular matrix ring R (see Theorem 2.12).

The structure of this paper is as follows: definitions, statements of results and examples are given in Section 2 and the proofs are collected in Section 3.

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2 Statements and Examples

Let us first make more explicit the universal property of localization:

Definition 2.1. A ring morphism $R \to R'$ is called σ-inverting if

$$\mathrm{id} \otimes \sigma : R' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \to R' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$$

is an isomorphism. The universal localization $i_\sigma : R \to \sigma^{-1}R$ is the initial object in the category of σ-inverting ring morphisms $R \to R'$. In other words, every σ-inverting ring morphism $R \to R'$ factors uniquely as a composite $R \to \sigma^{-1}R \to R'$.

Definition 2.2. An (A, M, B)-ring (S, f_A, f_M, f_B) is a ring S together with ring morphisms $f_A : A \to S$ and $f_B : B \to S$ and an (A, B)-bimodule morphism $f_M : M \to S$.

$$\begin{array}{ccccc} A & \xrightarrow{f_A} & S & \xleftarrow{f_B} & B \\ & & {\scriptstyle f_M}\uparrow & & \\ & & M & & \end{array}$$

It is understood that the (A, B)-bimodule structure on S is induced by f_A and f_B, so that $f_A(a)f_M(m) = f_M(am)$ and $f_M(m)f_B(b) = f_M(mb)$ for all $a \in A$, $b \in B$ and $m \in M$.

A morphism $(S, f_A, f_M, f_B) \to (S', f'_A, f'_M, f'_B)$ of (A, M, B)-rings is a ring morphism $\theta : S \to S'$ such that i) $\theta f_A = f'_A$, ii) $\theta f_M = f'_M$ and iii) $\theta f_B = f'_B$.

Definition 2.3. Suppose $p \in M$. Let $(T(M, p), \rho_A, \rho_M, \rho_B)$ denote the initial object in the subcategory of (A, M, B)-rings with the property $\rho_M(p) = 1$. For brevity we often write $T = T(M, p)$.

The ring T can be explicitly described in terms of generators and relations as follows. We have one generator x_m for each element $m \in M$ and relations:

(+) $x_m + x_{m'} = x_{m+m'}$

(a) $x_{ap}x_m = x_{am}$

(b) $x_m x_{pb} = x_{mb}$

(id) $x_p = 1$

for all $m, m' \in M$, $a \in A$ and $b \in B$. The morphisms ρ_A, ρ_M, ρ_B are

$$\begin{aligned}
\rho_A &: A \to T; a \mapsto x_{ap} \\
\rho_B &: B \to T; b \mapsto x_{pb} \\
\rho_M &: M \to T; m \mapsto x_m
\end{aligned}$$

Suppose $\sigma : \begin{pmatrix} A \\ 0 \end{pmatrix} \to \begin{pmatrix} M \\ B \end{pmatrix}$ is a morphism of left R-modules. We may write $\sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}$ for some $p \in M$. Let $T = T(M, p)$.

Theorem 2.4. *The universal localization $R \to \sigma^{-1}R$ is (isomorphic to)*

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} \rho_A & \rho_M \\ 0 & \rho_B \end{smallmatrix}\right)} \begin{pmatrix} T & T \\ T & T \end{pmatrix}.$$

Example 2.5.

1. Suppose $A = B = M$ and multiplication in A defines the (A, A)-bimodule structure on M. If $p = 1$ then $T = A$ and $\rho_A = \rho_M = \rho_B = \mathrm{id}_A$.
2. Suppose $A = B$ and $M = A \oplus A$ with the obvious bimodule structure. If $p = (1, 0)$ then T is the polynomial ring $A[x]$ in a central indeterminate x. The map $\rho_A = \rho_B$ is the inclusion of A in $A[x]$ while $\rho_M(1, 0) = 1$ and $\rho_M(0, 1) = x$.

The universal localizations corresponding to Example 2.5 are

1. $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \to \begin{pmatrix} A & A \\ A & A \end{pmatrix}$;
2. $\begin{pmatrix} A & A \oplus A \\ 0 & A \end{pmatrix} \to \begin{pmatrix} A[x] & A[x] \\ A[x] & A[x] \end{pmatrix}$.

Remark 2.6. One can regard the triangular matrix rings in these examples as path algebras over A for the quivers

$$1.\ \bullet \longrightarrow \bullet \qquad 2.\ \bullet \rightrightarrows \bullet$$

The universal localizations $R \to \sigma^{-1}R$ are obtained by introducing an inverse to the arrow in 1. and by introducing an inverse to one of the arrows in 2. See for example Benson [2, p.99] for an introduction to quivers.

The following examples subsume these:

Example 2.7. 1. (Amalgamated free product; Schofield [17, Thm. 4.10]) Suppose $i_A : C \to A$ and $i_B : C \to B$ are ring morphisms and $M = A \otimes_C B$. If $p = 1 \otimes 1$ then T is the amalgamated free product $A \underset{C}{\sqcup} B$ and appears in the pushout square

$$\begin{tikzcd} C \arrow[r, "i_A"] \arrow[d, "i_B"'] & A \arrow[d, "\rho_A"] \\ B \arrow[r, "\rho_B"'] & T \end{tikzcd}$$

The map ρ_M is given by $\rho_M(a \otimes b) = \rho_A(a)\rho_B(b)$ for all $a \in A$ and $b \in B$. We recover part 1. of Example 2.5 by setting $A = B = C$ and $i_A = i_B = \mathrm{id}$.

2. (HNN extension) Suppose $A = B$ and $i_1, i_2 : C \to A$ are ring morphisms. Let $A \otimes_C A$ denote the tensor product with C acting via i_1 on the first copy of A and by i_2 on the second copy. Let $M = A \oplus (A \otimes_C A)$ and $p = (1, 0 \otimes 0)$. Now $T = A *_C \mathbb{Z}[x]$ is generated by the elements in A together with an indeterminate x and has the relations in A together with $i_1(c)x = xi_2(c)$ for each $c \in C$. We have $\rho_A(a) = \rho_B(a) = a$ for all $a \in A$ while $\rho_M(1, 0 \otimes 0) = 1$ and $\rho_M(0, a_1 \otimes a_2) = a_1 x a_2$. If $C = A$ and $i_1 = i_2 = \mathrm{id}_A$ we recover part 2. of Example 2.5.

The following example is Theorem 13.1 of Schofield [17] and generalizes Example 2.7.

Example 2.8. 1. Suppose p generates M as a bimodule, i.e. $M = ApB$. Now T is generated by the elements of A and the elements of B subject to the relation $\sum_{i=1}^n a_i b_i = 0$ if $\sum_{i=1}^n a_i p b_i = 0$ (with $a_i \in A$ and $b_i \in B$). This ring T is denoted $A \underset{(M,p)}{\sqcup} B$ in [17, Ch13]. The maps ρ_A and ρ_B are obvious and ρ_M sends $\sum_i a_i p b_i$ to $\sum_i a_i b_i$.

2. Suppose $M = ApB \oplus N$ for some (A, B)-bimodule N. Now T is the tensor ring over $A \underset{(M,p)}{\sqcup} B$ of

$$(A \underset{(M,p)}{\sqcup} B) \otimes_A N \otimes_B (A \underset{(M,p)}{\sqcup} B).$$

We may vary the choice of p as the following example illustrates:

Example 2.9. Suppose $A = B = M = \mathbb{Z}$ and $p = 2$. In this case $T = \mathbb{Z}\left[\frac{1}{2}\right]$ and $\rho_A = \rho_B$ is the inclusion of $\mathbb{Z}$ in $\mathbb{Z}\left[\frac{1}{2}\right]$ while $\rho_M(n) = n/2$ for all $n \in \mathbb{Z}$.

Example 2.9 can be verified by direct calculation using Theorem 2.4 or deduced from part 1. of Example 2.5 by setting $a_0 = b_0 = 2$ in the following more general proposition. Before stating it, let us remark that the universal property of $T = T(M, p)$ implies that $T(M, p)$ is functorial in (M, p). An (A, B)-bimodule morphism $\phi : M \to M'$ with $\phi(p) = p'$ induces a ring morphism $T(M, p) \to T(M', p')$.

Proposition 2.10. *Suppose A and B are rings, M is an (A,B)-bimodule and $p \in M$. If $a_0 \in A$ and $b_0 \in B$ satisfy $a_0 m = m b_0$ for all $m \in M$ then*

1. *The element $\rho_M(a_0 p) = x_{a_0 p} = x_{p b_0}$ is central in $T(M,p)$.*

2. *The ring morphism $\phi : T(M,p) \to T(M, a_0 p) = T(M, p b_0)$ induced by the bimodule morphism $\phi : M \to M; m \mapsto a_0 m = m b_0$ is the universal localization of $T(M,p)$ making invertible the element $x_{a_0 p}$.*

Since $x_{a_0 p}$ is central each element in $T(M, a_0 p)$ can be written as a fraction α/β with numerator $\alpha \in T(M,p)$ and denominator $\beta = x^r_{a_0 p}$ for some non-negative integer r.

Having described universal localization of the ring R in Theorem 2.4 we may also describe the universal localization $\sigma^{-1}R \otimes_R N$ of a left R-module N. For the convenience of the reader let us first recall the structure of modules over a triangular matrix ring.

Lemma 2.11. *Every left R-module N can be written canonically as a triple*

$$(N_A, N_B, f : M \otimes_B N_B \to N_A)$$

where N_A is a left A-module, N_B is a left B-module and f is a morphism of left A-modules.

A proof of this lemma is included in Section 3 below. Localization of modules can be expressed as follows:

Theorem 2.12. [2] *For any left R-module $N = (N_A, N_B, f)$ the localization left $\sigma^{-1}R$-module $\sigma^{-1}N = \sigma^{-1}R \otimes_R N$ is isomorphic to $\begin{pmatrix} L \\ L \end{pmatrix}$ with $\sigma^{-1}R = M_2(T)$, $T = T(M,p)$, L the left T-module defined by*

$$\begin{aligned} L &= (T\ T) \otimes_R N \\ &= \operatorname{coker}\Big(\begin{pmatrix} 1 \otimes f \\ g \otimes 1 \end{pmatrix} : T \otimes_A M \otimes_B N_B \to (T \otimes_A N_A) \oplus (T \otimes_B N_B)\Big) \end{aligned}$$

with g the (T,B)-bimodule morphism

$$g \ : \ T \otimes_A M \to T \ ; \ t \otimes m \mapsto -t x_m \ ,$$

and $M_2(T)$ acting on the left of $\begin{pmatrix} L \\ L \end{pmatrix}$ by matrix multiplication

3 Proofs

The remainder of this paper is devoted to the proofs of Theorem 2.4, Proposition 2.10 and Theorem 2.12.

[2]This corrects Theorem 2.12 in the preprint version arXiv:math.RA/0407407.

3.1 Localization as Pushout

Before proving Theorem 2.4 we show that there is a pushout diagram

$$\begin{tikzcd} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \arrow[r] \arrow[d, "\alpha"'] & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \arrow[d] \\ R \arrow[r] & \sigma^{-1}R \end{tikzcd}$$

where $\alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\alpha\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$. Bergman observed [4, p.71] that more generally, up to Morita equivalence every localization $R \to \sigma^{-1}R$ appears in such a pushout diagram.

It suffices to check that the lower horizontal arrow in any pushout

$$\begin{tikzcd} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \arrow[r] \arrow[d, "\alpha"'] & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \arrow[d, "\theta"'] \\ R \arrow[r, "i"'] & S \end{tikzcd}$$

is i) σ-inverting and ii) universal among σ-inverting ring morphisms. The universal property of a pushout will be shown to be the universal property of a universal localization, so that such a commutative diagram is a pushout if and only if S is a universal localization $\sigma^{-1}R$.

i) The map $\mathrm{id} \otimes \sigma : S \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \to S \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$ has inverse given by the composite

$$S \otimes_R \begin{pmatrix} M \\ B \end{pmatrix} \subset S \otimes_R R \cong S \xrightarrow{\gamma} S \cong S \otimes_R R \twoheadrightarrow S \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where γ multiplies on the right by $\theta\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

ii) If $i' : R \to S'$ is a σ-inverting ring morphism then there is an inverse $\psi : S' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix} \to S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix}$ to $\mathrm{id} \otimes \sigma$. It is argued shortly below that there is a (unique) diagram

$$\begin{tikzcd} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \arrow[r] \arrow[d, "\alpha"'] & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \arrow[d, "\theta"'] \arrow[ddr, bend left, "\theta'"] & \\ R \arrow[r, "i"] \arrow[drr, bend right, "i'"'] & S \arrow[dr, dotted] & \\ & & S' \end{tikzcd} \tag{1}$$

where θ' sends $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to $\psi\left(1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \in S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \subset S'$. Since S is a pushout there is a unique morphism $S \to S'$ to complete the diagram and so i' factors uniquely through i.

To show uniqueness of (1), note that in S' multiplication on the right by $\theta'\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ must coincide with the morphism

$$\begin{pmatrix} 0 & 0 \\ \mathrm{id} \otimes \sigma & 0 \end{pmatrix} : S' \otimes \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes \begin{pmatrix} M \\ B \end{pmatrix} \longrightarrow S' \otimes \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes \begin{pmatrix} M \\ B \end{pmatrix}$$

so multiplication on the right by $\theta'\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ coincides with $\begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix}$. Now $1 \in S'$ may be written

$$\left(1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} , 1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \in S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$$

so $\theta'\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \psi\left(1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. The reader may verify that this formula demonstrates the existence of a commutative diagram (1).

3.2 Identifying $\sigma^{-1}R$

Proof of Theorem 2.4. It suffices to show that the diagram of ring morphisms

$$\begin{tikzcd}
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \arrow[r] \arrow[d, "\alpha"'] & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \arrow[d] \\
\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \arrow[r, "\rho"'] & \begin{pmatrix} T & T \\ T & T \end{pmatrix}
\end{tikzcd}$$

is a pushout, where $T = T(M,p)$, $\rho = \begin{pmatrix} \rho_A & \rho_M \\ 0 & \rho_B \end{pmatrix}$ and α is defined as in Section 3.1. Given a diagram of ring morphisms

$$\begin{tikzcd}
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \arrow[r] \arrow[d, "\alpha"'] & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \arrow[d] \arrow[ddr, bend left, "\theta"] & \\
\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \arrow[r, "\rho"'] \arrow[drr, bend right, "\rho'"'] & \begin{pmatrix} T & T \\ T & T \end{pmatrix} \arrow[dr, dotted, "\gamma"] & \\
& & S
\end{tikzcd} \tag{2}$$

we must show that there is a unique morphism γ to complete the diagram. The map θ induces a decomposition of S as a matrix ring $M_2(S') = \begin{pmatrix} S' & S' \\ S' & S' \end{pmatrix}$ with S' the centralizer of $\theta(M_2(\mathbb{Z})) \subset S$. In particular, $\theta(e_{ij}) = e_{ij}$ for $i, j \in \{1, 2\}$. Any morphism γ which makes the diagram commute must be of the form $\gamma = M_2(\gamma')$ for some ring morphism $\gamma' : T \to S'$ (e.g. Cohn [9, p.1] or Lam [15, (17.7)]). Commutativity of the diagram implies that ρ' also respects the 2×2 matrix structure and we may write

$$\rho' = \begin{pmatrix} \rho'_A & \rho'_M \\ 0 & \rho'_B \end{pmatrix} : \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \longrightarrow \begin{pmatrix} S' & S' \\ S' & S' \end{pmatrix}$$

with $\rho'_M(p) = 1$ as one sees by considering the images of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ under the maps in the diagram (2) above. Since ρ' is a ring morphism, one finds

$$\begin{pmatrix} \rho'_A(aa') & \rho'_M(am' + mb') \\ 0 & \rho'_B(bb') \end{pmatrix} = \begin{pmatrix} \rho'_A(a)\rho'_A(a') & \rho'_A(a)\rho'_M(m') + \rho'_M(m)\rho'_B(b') \\ 0 & \rho'_B(b)\rho'_B(b') \end{pmatrix}$$

for all $a, a' \in A$, $b, b' \in B$ and $m, m' \in M$. Hence the maps $\rho'_A : A \to S'$ and $\rho'_B : B \to S'$ are ring morphisms and ρ'_M is a morphism of (A, B)-bimodules. Thus S' is an (A, M, B)-ring with respect to the maps $\rho'_A, \rho'_M, \rho'_B$ such that $\rho'_M(p) = 1$. By the universal property of T there exists a unique morphism $\gamma' : T \to S'$ such that $M_2(\gamma') : M_2(T) \to M_2(S') = S$ completes the diagram (2) above. □

Proof of Proposition 2.10. 1. In $T(M, p)$ we have $x_{a_0 p} x_m = x_{a_0 m} = x_{m b_0} = x_m x_{p b_0} = x_m x_{a_0 p}$ for all $m \in M$.
2. The map $\phi : M \to M; m \mapsto a_0 m$ induces

$$\begin{aligned} \phi : T(M, p) &\to T(M, a_0 p) \\ x_m &\mapsto x_{a_0 m} \end{aligned} \tag{3}$$

In particular $\phi(x_{a_0 p}) = x_{a_0^2 p} \in T(M, a_0 p)$ and we have

$$x_{a_0^2 p} x_p = x_{a_0 (a_0 p)} x_p = x_{a_0 p} = 1 = x_{p b_0} = x_p x_{p b_0^2} = x_p x_{a_0^2 p}$$

so $\phi(x_{a_0 p})$ is invertible.

We must check that (3) is universal. If $f : T(M, p) \to S$ is a ring morphism and $f(x_{a_0 p})$ is invertible, we claim that there exists a unique $\widetilde{f} : T(M, a_0 p) \to S$ such that $\widetilde{f}\phi = f$.
Uniqueness: Suppose $\widetilde{f}\phi = f$. For each $m \in M$ we have

$$\widetilde{f}(x_{a_0 m}) = \widetilde{f}\phi(x_m) = f(x_m).$$

Now $f(x_{a_0 p})\widetilde{f}(x_m) = \widetilde{f}\phi(x_{a_0 p})\widetilde{f}(x_m) = \widetilde{f}(x_{a_0 (a_0 p)} x_m) = \widetilde{f}(x_{a_0 m}) = f(x_m)$ so

$$\widetilde{f}(x_m) = (f(x_{a_0 p}))^{-1} f(x_m). \tag{4}$$

Existence: It is straightforward to check that equation (4) provides a definition of $\widetilde{f}$ which respects the relations (+),(a),(b) and (id) in $T(M, a_0p)$. Relation (b), for example, is proved by the equations

$$\widetilde{f}(x_m)\widetilde{f}(x_{a_0pb}) = f(x_{a_0p})^{-1}f(x_m)f(x_{pb}) = f(x_{a_0p})^{-1}f(x_{mb}) = \widetilde{f}(x_{mb})$$

and the other relations are left to the reader. □

3.3 Module Localization

We turn finally to the universal localization $\sigma^{-1}R \otimes_R N$ of an R-module N.

Proof of Lemma 2.11. If N is a left R-module, set $N_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} N$ and set $N_B = N/N_A$. If $m \in M$ and $n_B \in N_B$ choose a lift $x \in N$ and define the map $f : M \otimes N_B \to N_A$ by $f(m \otimes n_B) = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} x$. Conversely, given a triple (N_A, N_B, f) one recovers a left R-module $\begin{pmatrix} N_A \\ N_B \end{pmatrix}$ with

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} n_A \\ n_B \end{pmatrix} = \begin{pmatrix} an_A + f(m \otimes n_B) \\ bn_B \end{pmatrix}$$

for all $a \in A$, $b \in B$, $m \in M$, $n_A \in N_A$, $n_B \in N_B$. □

Proof of Theorem 2.12. As in the statement, let $T = T(M, p)$ and define the left T-module

$$L \;=\; \operatorname{coker}\Big(\begin{pmatrix} 1 \otimes f \\ g \otimes 1 \end{pmatrix} : T \otimes_A M \otimes_B N_B \to (T \otimes_A N_A) \oplus (T \otimes_B N_B) \Big) \;.$$

We shall establish an isomorphism of left T-modules

$$\begin{pmatrix} T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \;\cong\; L \qquad (5)$$

and leave to the reader the straightforward deduction that there is an isomorphism of $\sigma^{-1}R$-modules

$$\sigma^{-1}R \otimes_R N = \begin{pmatrix} T & T \\ T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \cong \begin{pmatrix} L \\ L \end{pmatrix}.$$

The left T-module morphism

$$\alpha : L \to \begin{pmatrix} T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \;;$$
$$(t \otimes n_A, t' \otimes n_B) \mapsto \begin{pmatrix} t & 0 \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & t' \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix}$$

is well-defined, since

$$
\begin{aligned}
&\alpha(t \otimes_A f(m,n_B), g(t,m) \otimes_B n_B) \\
&\qquad = \alpha(t \otimes_A f(m,n_B), -tx_m \otimes_B n_B) \\
&\qquad = \begin{pmatrix} t & 0 \end{pmatrix} \otimes_R \begin{pmatrix} f(m,n_B) \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & tx_m \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} \\
&\qquad = \begin{pmatrix} t & 0 \end{pmatrix} \otimes_R \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ n_B \end{pmatrix} - \begin{pmatrix} t & 0 \end{pmatrix} \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} \\
&\qquad = 0 \in \begin{pmatrix} T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} .
\end{aligned}
$$

The left T-module morphism

$$
\beta : \begin{pmatrix} T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \to L \ ; \ \begin{pmatrix} t & t' \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix} \mapsto (t \otimes n_A, t' \otimes n_B)
$$

is well-defined, since

$$
\begin{aligned}
&\beta\Big(\begin{pmatrix} t & t' \end{pmatrix} \otimes_R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} n_A \\ n_B \end{pmatrix} - \begin{pmatrix} t & t' \end{pmatrix} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix} \Big) \\
&\quad = (t \otimes (an_A + f(m,n_b)), t' \otimes bn_B) - (ta \otimes n_A, (tx_m + t'b) \otimes n_B) \\
&\quad = (t \otimes f(m,n_B), -tx_m \otimes n_B) \\
&\quad = (1 \otimes f, g \otimes 1)(t \otimes m \otimes n_B) \ = \ 0 \in L .
\end{aligned}
$$

It is immediate that $\beta\alpha = \mathrm{id}$. To prove (5) we must check that $\alpha\beta = \mathrm{id}$ or in other words that

$$
\begin{pmatrix} t & t' \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix} = \begin{pmatrix} t & 0 \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & t' \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} .
$$

This equation follows from the following two calculations:

$$
\begin{aligned}
&\begin{pmatrix} t & 0 \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} = \begin{pmatrix} t & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes_R \begin{pmatrix} 0 \\ n_B \end{pmatrix} = 0 \ ; \\
&\begin{pmatrix} 0 & t' \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & t' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ 0 \end{pmatrix} = 0 \ .
\end{aligned}
$$

□

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